Online Appendix (Not intended for Publication): Federal Reserve Credibility and the Term Structure of Interest Rates

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1 Derivation of Bond Yields

1.1 No regime switching

The rational expectation solution to the DSGE model can be described by the following equation:

$$\begin{aligned} \xi_t &= F\xi_{t-1} + Gv_t && (1) \\ v_t &\sim N(0, Q) && (2) \end{aligned}$$

Without loss of generality we can write the log of the stochastic discount factor m_{t+1} as,

$$m_{t+1} = -\lambda_0 - \lambda_1'\xi_{t+1} - \lambda_2'\xi_t \tag{3}$$

Let $P_t^{(n)}$ denote the price of a n-period zero-coupon bond at time t. From the no-arbitrage asset pricing equations

$$P_t^{(n)} = E_t\left[M_{t+1}P_{t+1}^{(n-1)}\right] \tag{4}$$

With the joint lognormal distributions we can write this as (lowercase letters denote logs)

$$P_t^{(n)} = exp\left\{E_t m_{t+1} + E_t p_{t+1}^{(n-1)} + \frac{1}{2}Var_t m_{t+1} + \frac{1}{2}Var_t p_{t+1}^{(n-1)} + Cov_t(m_{t+1}, p_{t+1}^{(n-1)})\right\} \tag{5}$$

using the property that if a random variable $X \sim logN(\mu, \sigma^2)$ then $logX \sim (\mu, \sigma^2)$. Moreover, $E[X] = exp\{\mu + (1/2)\sigma^2\} = exp\{E(logX) + (1/2)Var(logX)\}$ Assuming an affine structure for log bond prices gives

$$p_t^{(n)} = -A_n - B_n'\xi_t \tag{6}$$

Using the above equations we can derive

$$\begin{aligned} E_t m_{t+1} &= -\lambda_0 - \lambda_1' F\xi_t - \lambda_2'\xi_t \\ E_t p_{t+1}^{(n-1)} &= -A_{n-1} - B_{n-1}'F\xi_t \\ Var_t m_{t+1} &= \lambda_1' GQG'\lambda_1 \\ Var_t p_{t+1}^{(n-1)} &= B_{n-1}'GQG'B_{n-1} \\ Cov_t(m_{t+1}, p_{t+1}^{(n-1)}) &= \lambda_1'GQG'B_{n-1} \end{aligned}$$

Taking log of equation (5) and plugging in the above gives

$$\begin{gathered} -A_n - B_n'\xi_t = -\lambda_0 - \lambda_1'F\xi_t - \lambda_2'\xi_t - A_{n-1} - B_{n-1}'F\xi_t \\ +\tfrac{1}{2}\left(\lambda_1'GQG'\lambda_1\right) + \tfrac{1}{2}\left(B_{n-1}'GQG'B_{n-1}\right) + \lambda_1'GQG'B_{n-1} \end{gathered}$$

Matching coefficients on the constant and noting that $\lambda_1'GQG'B_{n-1} = B_{n-1}'GQG'\lambda_1 = \frac{1}{2}\left(\lambda_1'GQG'B_{n-1} + B_{n-1}'GQ\right.$

$$\begin{aligned} A_n &= \lambda_0 + A_{n-1} - \frac{1}{2}\left(\lambda_1'GQG'\lambda_1\right) - \frac{1}{2}\left(B_{n-1}'GQG'B_{n-1}\right) - \lambda_1'GQG'B_{n-1} && (7) \\ &= \lambda_0 + A_{n-1} - \frac{1}{2}\left(\lambda_1'GQG'\lambda_1\right) - \frac{1}{2}\left(B_{n-1}'GQG'B_{n-1}\right) - \frac{1}{2}\left(\lambda_1'GQG'B_{n-1}\right) - \frac{1}{2}\left(B_{n-1}'GQG'\lambda_1\right) && (8) \\ &= \lambda_0 + A_{n-1} - \frac{1}{2}(\lambda_1 + B_{n-1})'GQG'(\lambda_1 + B_{n-1}) && (9) \end{aligned}$$

Matching coefficients on ξ_t we get

$$B_n = F'B_{n-1} + F'\lambda_1 + \lambda_2 \tag{10}$$

1.2 Regime switching: Loose commitment and regime switching shock variances

In this case the dynamics of state variables can be described by the following equation:

$$\xi_t = F_{s_t}\xi_{t-1} + Gv_t \tag{11}$$
$$v_t \sim N(0, Q_{s_t^v}) \tag{12}$$

where s_t (with transition matrix P) governs the loose commitment switches and s_t^v (with transition matrix P^v) governs the switches in the shock variance, each being a two-state Markov chain. Let $\widetilde{s}_t$ be a composite regime indicator with 4 regimes and a transition matrix $\widetilde{P} = P \otimes P^v$

Now consider the same setup for the stochastic discount factor

$$m_{t+1} = -\lambda_0 - \lambda_1'\xi_{t+1} - \lambda_2'\xi_t \tag{13}$$

Let $P_{n,t}$ denote the price of a n-period zero-coupon bond at time t. It then follows that, for $n \geq 0$,

$$P_{n,t} = e^{-A_{n,\widetilde{s}_t} - B'_{n,\widetilde{s}_t}\xi_t} \tag{14}$$

The coefficients $A_{n,\widetilde{s}_t}$ and $B_{n,\widetilde{s}_t}$ are allowed to depend on the regime indicator $\widetilde{s}_t$. Now consider p_{t+1}^{n-1}, conditional on $\widetilde{s}_{t+1}$, the mean and variance of the log bond price can be written as

$$E\left[m_{t+1}|I_{t+1}\right] = -\lambda_0 - \lambda_1' F_{\widetilde{s}_{t+1}}\xi_t - \lambda_2'\xi_t \tag{15}$$
$$Var\left[m_{t+1}|I_{t+1}\right] = \lambda_1' G Q_{\widetilde{s}_{t+1}} G'\lambda_1 \tag{16}$$
$$E\left[p_{t+1}^{(n-1)}|I_{t+1}\right] = -A_{n-1,\widetilde{s}_{t+1}} - B_{n-1,\widetilde{s}_{t+1}} F_{\widetilde{s}_{t+1}}\xi_t \tag{17}$$
$$Var\left[p_{t+1}^{(n-1)}|I_{t+1}\right] = B'_{n-1,\widetilde{s}_{t+1}} G Q_{\widetilde{s}_{t+1}} G' B_{n-1,\widetilde{s}_{t+1}} \tag{18}$$
$$Cov\left[(m_{t+1}, p_{t+1}^{(n-1)})|I_{t+1}\right] = \lambda_1' G Q_{\widetilde{s}_{t+1}} G' B_{n-1} \tag{19}$$

where Ω_t is all available information upto time t and $I_{t+1} = \Omega_t \cup \widetilde{s}_{t+1}$.

$$P_t^{(n)} = E\left[M_{t+1}P_{t+1}^{(n-1)}|\Omega_t\right] \tag{20}$$
$$= \sum_{\widetilde{s}_{t+1}} \pi_{i,\widetilde{s}_{t+1}} E\left[M_{t+1}P_{t+1}^{(n-1)}|I_{t+1}\right] \tag{21}$$
$$= \sum_{\widetilde{s}_{t+1}} \pi_{i,\widetilde{s}_{t+1}} exp\{E\left[m_{t+1}|I_{t+1}\right] + E\left[p_{t+1}^{(n-1)}|I_{t+1}\right] + \frac{1}{2}Var\left[m_{t+1}|I_{t+1}\right] + Cov\left[(m_{t+1}, p_{t+1}^{(n-1)})|I_{t+1}\right] \tag{22}$$

Following Bansal & Zhou (2002), use the log-linear approximation $exp(x) \approx 1 + x$. The RHS of the above equation becomes

$$\sum_{\widetilde{s}_{t+1}} \pi_{i,s_{t+1}} \left(E\left[m_{t+1}|I_{t+1}\right] + E\left[p_{t+1}^{(n-1)}|I_{t+1}\right] + \frac{1}{2}Var\left[m_{t+1}|I_{t+1}\right] + Cov\left[(m_{t+1}, p_{t+1}^{(n-1)})|I_{t+1}\right] + 1\right) \tag{23}$$

Moving the 1 on the other side, the LHS becomes

$$= P_t^{(n)} - 1 \tag{24}$$
$$= e^{-A_{n,\widetilde{s}_t} - B'_{n,\widetilde{s}_t}\xi_t} - 1 \tag{25}$$
$$\approx -A_{n,\widetilde{s}_t} - B'_{n,\widetilde{s}_t}\xi_t \tag{26}$$

Plugging in equations (15-19) into the RHS we get

$$\sum_{\widetilde{s}_{t+1}} \pi_{i,\widetilde{s}_{t+1}}(-\lambda_0 - \lambda_1' F_{\widetilde{s}_{t+1}}\xi_t - \lambda_2'\xi_t - A_{n-1,\widetilde{s}_{t+1}} - B_{n-1,\widetilde{s}_{t+1}} F_{\widetilde{s}_{t+1}}\xi_t$$

$$+\frac{1}{2}\lambda_1' GQ_{\widetilde{s}_{t+1}} G'\lambda_1 + \frac{1}{2}B'_{n-1,\widetilde{s}_{t+1}} GQ_{s_{t+1}} G' B_{n-1,\widetilde{s}_{t+1}} + \lambda_1' GQ_{\widetilde{s}_{t+1}} G' B_{n-1})$$

Matching coefficients we get

$$\begin{pmatrix} A_{n,\widetilde{s}_t=1} \\ A_{n,\widetilde{s}_t=2} \\ A_{n,\widetilde{s}_t=3} \\ A_{n,\widetilde{s}_t=4} \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{pmatrix} \begin{pmatrix} \lambda_0 + A_{n-1,\widetilde{s}_{t+1}=1} - \frac{1}{2}(\lambda_1 + B_{n-1,s_{t+1}=1})' GQ_{\widetilde{s}_{t+1}=1} G'(\lambda_1 + B_{n-1,s_{t+1}=1}) \\ \lambda_0 + A_{n-1,\widetilde{s}_{t+1}=2} - \frac{1}{2}(\lambda_1 + B_{n-1,s_{t+1}=2})' GQ_{\widetilde{s}_{t+1}=2} G'(\lambda_1 + B_{n-1,s_{t+1}=2}) \\ \lambda_0 + A_{n-1,\widetilde{s}_{t+1}=3} - \frac{1}{2}(\lambda_1 + B_{n-1,s_{t+1}=3})' GQ_{\widetilde{s}_{t+1}=3} G'(\lambda_1 + B_{n-1,s_{t+1}=3}) \\ \lambda_0 + A_{n-1,\widetilde{s}_{t+1}=4} - \frac{1}{2}(\lambda_1 + B_{n-1,s_{t+1}=4})' GQ_{\widetilde{s}_{t+1}=4} G'(\lambda_1 + B_{n-1,s_{t+1}=4}) \end{pmatrix} \tag{27}$$

$$\begin{pmatrix} B_{n,\widetilde{s}_t=1} \\ B_{n,\widetilde{s}_t=2} \\ B_{n,\widetilde{s}_t=3} \\ B_{n,\widetilde{s}_t=4} \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{pmatrix} \begin{pmatrix} F'_{\widetilde{s}_{t+1}=1} B_{n-1,\widetilde{s}_{t+1}=1} + F'_{\widetilde{s}_{t+1}=1}\lambda_1 + \lambda_2 \\ F'_{\widetilde{s}_{t+1}=2} B_{n-1,\widetilde{s}_{t+1}=2} + F'_{\widetilde{s}_{t+1}=2}\lambda_1 + \lambda_2 \\ F'_{\widetilde{s}_{t+1}=3} B_{n-1,\widetilde{s}_{t+1}=3} + F'_{\widetilde{s}_{t+1}=3}\lambda_1 + \lambda_2 \\ F'_{\widetilde{s}_{t+1}=4} B_{n-1,\widetilde{s}_{t+1}=4} + F'_{\widetilde{s}_{t+1}=4}\lambda_1 + \lambda_2 \end{pmatrix} \tag{28}$$

Note for the specific setup here the transition matrices are given as follows.

$$P = \begin{bmatrix} \gamma & 1-\gamma \\ \gamma & 1-\gamma \end{bmatrix}$$

$$P^v = \begin{bmatrix} p_1 & (1-p_1) \\ (1-p_2) & p_2 \end{bmatrix}$$

The composite regime switching variable is $\widetilde{s}_t$, noting that s_t and s_t^v are independent.

$$\widetilde{s}_t = 1 \text{ if } s_t = 1 \text{ and } s_t^v = 1$$
$$\widetilde{s}_t = 2 \text{ if } s_t = 1 \text{ and } s_t^v = 2$$
$$\widetilde{s}_t = 3 \text{ if } s_t = 0 \text{ and } s_t^v = 1$$
$$\widetilde{s}_t = 4 \text{ if } s_t = 0 \text{ and } s_t^v = 2$$

1.3 Regime switching: Loose commitment

In this case the dynamics of state variables can be described by the following equation:

$$\xi_t = F_{s_t}\xi_{t-1} + Gv_t \tag{29}$$
$$v_t \sim N(0, Q) \tag{30}$$

where s_t governs the loose commitment switches with transition matrix P depending on the probability of commitment γ

$$P^m = \begin{bmatrix} \gamma & 1-\gamma \\ \gamma & 1-\gamma \end{bmatrix}$$

Thus probability that next period $s_t = 1$ (continuing plans) is γ regardless of the current state. Similarly probability that next period $s_t = 0$ (re-optimization) is $1 - \gamma$. Following the same procedure above we can derive

$$\begin{pmatrix} A_{n,s_t=1} \\ A_{n,s_t=0} \end{pmatrix} = \begin{pmatrix} \gamma & 1-\gamma \\ \gamma & 1-\gamma \end{pmatrix} \begin{pmatrix} \lambda_0 + A_{n-1,s_{t+1}=1} - \frac{1}{2}(\lambda_1 + B_{n-1,s_{t+1}=1})' G Q_{s_{t+1}=1} G' (\lambda_1 + B_{n-1,s_{t+1}=1}) \\ \lambda_0 + A_{n-1,s_{t+1}=0} - \frac{1}{2}(\lambda_1 + B_{n-1,s_{t+1}=0})' G Q_{s_{t+1}=0} G' (\lambda_1 + B_{n-1,s_{t+1}=0}) \end{pmatrix} \tag{31}$$

$$\begin{pmatrix} B_{n,s_t=1} \\ B_{n,s_t=0} \end{pmatrix} = \begin{pmatrix} \gamma & 1-\gamma \\ \gamma & 1-\gamma \end{pmatrix} \begin{pmatrix} F'_{s^m_{t+1}=1} B_{n-1,s_{t+1}=1} + F'_{s_{t+1}=1}\lambda_1 + \lambda_2 \\ F'_{s^m_{t+1}=0} B_{n-1,s_{t+1}=0} + F'_{s_{t+1}=0}\lambda_1 + \lambda_2 \end{pmatrix} \tag{32}$$

Since the rows of the transition matrix are the same, we conclude that $A_{n,s_t=1} = A_{n,s_t=0}$ and $B_{n,s_t=1} = B_{n,s_t=0}$